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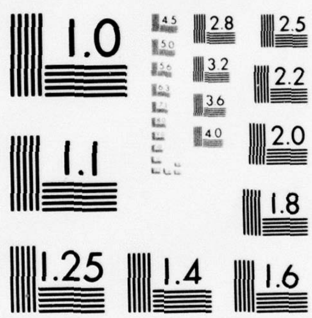

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SCHOOL OF ENGINEERING AND APPLIED SCIENCE

University of Virginia

Charlottesville, Virginia 22901

A Report

TRANSIENT RESPONSE OF CONTINUOUS ELASTIC STRUCTURES

by

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and

J. Strenkowski

Technical Report No. UVA/525303/ME77/101

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A general theory for the dynamic response of linear damped continuous structured members is formulated with a modal analysis. The theory applies to elastic or viscoelastic solids. Proportional and non-proportional damping are included.

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LIST OF SYMBOLS

a	constant of proportionality for viscous damping
A_j	N-square spatial matrix
b	constant of proportionality for viscous damping
B	bilinear functional
c	external or viscous damping coefficient
c_i	discrete dashpot coefficient
c_m	viscous proportionality constant, $c_m = a + \lambda_m^2 b$
$[C]$	bearing damping matrix
C_m	temporal coefficient
dA	differential area element
dV	differential volume element
D	N-square matrix linear differential operator
$\tilde{D}$	N-square algebraic adjoint matrix differential operator
$\tilde{D}^*$	N-square Hermitian adjoint matrix differential operator
D_{ij}	(i,j) element in the matrix differential operator D
E	modulus of elasticity
E_m	temporal coefficient
f	forcing function
f_m	temporal coefficient for classical systems; loading term for forced normal mode response
F	column vector of body forces
F_m	generalized forcing function
g_m	temporal coefficient for non-classical systems
$G(t)$	relaxation modulus for viscoelastic material
h_m	temporal coefficient

List of Symbols (continued)

$H(t)$	Heaviside unit function
H_m	generalized forcing function for structural member of viscoelastic material
i	$\sqrt{-1}$
I	moment of inertia taken about the neutral axis
$I_m\{S_n\}$	denotes imaginary part of S_n
$J(t)$	complex compliance of a viscoelastic material
k	Winkler (elastic) foundation modulus
k_i	extension spring constant
ℓ	length of beam
L_i	spatial matrix differential operator
M	bending moment
M_n	norm for forced normal mode response
N_m	Norm
P_k	viscoelastic material constant
P	temporal differential operator
P_i	column vectors of non-homogeneous boundary and in-span condition
q	applied loading intensity
q_m	temporal coefficient
q_k	viscoelastic material constant
Q	temporal differential operator; transverse shear force on a radial face
Q_m	classical norm
R_m	denotes right-hand side of viscoelastic equation
$Re\{S_n\}$	denotes the real part of S_n

List of Symbols (continued)

s	Laplace transform variable
S	denotes bounding surface and in-span condition locations
S_m	damped frequency or complex eigenvalue
t	time
$[T_E]$	denotes transfer matrix for an elastic beam section
$[T_M]$	denotes a beam transfer matrix for a lumped mass and dashpot station
$u(x,t)$	column vector of dependent state variables
$u(a_1,t)$	prescribed transverse beam deflection at $x = a_1$
u_i	denotes i^{th} element of column vector $u(x,t)$
u_n	generic symbol used to represent ψ_n or v_n
v_n	column vector of damped (or non-classical) mode shapes or eigenfunctions corresponding to the n^{th} damped frequency
v_s	column vector of state variables representing the quási-static response
$\bar{v}_s$	column vector denoting the quasi-static response of a viscoelastic member
V	shear force
w	transverse beam deflection
x	general point in the multidimensional region; one-dimensional coordinate direction in a rectangular coordinate system

<u>Greek Symbols</u>

α	constant
β_n	λ_n^2
γ	structural damping factor
γ_n	$-\mathrm{Re}\{S_n\}$

List of Symbols (continued)

$\delta(x-a_i)$	delta function
δ_{mn}	Kronecker delta
$\Delta\ell_i$	length of section i
ε	constant for Voigt-Kelvin model
ξ_m	constant related to λ_m
η	dashpot constant in the Voigt-Kelvin model
η_m	temporal coefficient
θ	slope of beam
κ	q_0/q_1 in the Voigt-Kelvin model
λ_n	denotes the undamped (or classical) frequency or eigenvalue
μ_n	generic symbol used to represent λ_n (classical) or S_n (non-classical)
ν	Poisson's ratio
ξ_m	temporal coefficient
π	3.14159 ...
ρ	mass density
τ	dummy time variable
ψ_n	column vector of undamped (or classical) mode shapes or eigenfunctions corresponding to the nth frequency
ω_n	Im $\{S_n\}$
Ω	driving frequency for harmonic excitation

<u>Subscripts</u>

0	denotes initial conditions; initial spatial location
a_i	denotes $x = a_i$ location on beam
a_1	denotes $x = a_1$ location on beam

List of Symbols (continued)

i	denotes $x = x_i$ axial location
s	denotes the quasi-static response

Superscripts

*	complex conjugate
(k)	denotes kth derivative with respect to x
+	"just to the right of"
-	"just to the left of"
R	right
L	left
†	denotes quantities associated with the classical free motion problem

Overbars

~	denotes the corresponding adjoint quantities
-	denotes the Laplace transform

Mathematical Notation

$\frac{\partial}{\partial x}$	denotes partial derivative with respect to x
$\frac{d}{dx}$	denotes the ordinary derivative with respect to x
$\frac{\partial^{(j)}}{\partial t^{(j)}}$	denotes jth partial derivative with respect to t; also use ∂_j for conciseness
$\cdot$	$\frac{\partial}{\partial t}$
$\cdot\cdot$	$\partial^2/\partial t^2$

Introduction:

The purpose of this work is to formulate a general theory for the dynamic response of linear damped continuous structural members subjected to arbitrary excitation forces. Typical members which are included in this theory are rods, beams, plates, shells, and rotating shafts. The damping models to be considered include the traditional viscous dashpot, so-called hysteretic (or material or structural) damping, and viscoelastic material damping.

A general technique for finding the dynamic response is modal analysis, and it will be used exclusively in the following developments. This method has been employed extensively for both undamped discrete and undamped continuous linear systems (see for example, Hurty and Rubinstein [1][1] and Meirovitch [2]). However, when damping is present, the use of modal analysis has been largely restricted to self-adjoint systems of equations which may be uncoupled with the classical or undamped normal modes. Meirovitch [2] presents several examples of such systems which include structural members with proportional viscous and linear hysteretic damping. For viscoelastic structural members, Valanis [3] was the first to propose a suitable modal solution. By assuming that Poisson's ratio remains constant, the dynamic viscoelastic problem may be resolved into a quasi-static viscoelastic problem and a dynamic undamped elastic problem. Robertson and Thomas [4] used a similar technique which allowed more general boundary and non-zero initial conditions to be considered.

[1] Numbers in brackets refer to References at end of paper.

an example of this technique as applied to a self-adjoint viscoelastic Timoshenko beam is given by Robertson [5].

When other forms of damping, such as non-proportional viscous damping, are included in a structural member model the classical modes cannot uncouple the equations of motion. Hence, modal analysis in the classical sense may not be used to determine the response. However, for a discrete system with viscous damping, it has been shown by Foss [6] that a resolution of the response is possible by using a new set of orthogonal eigenfunctions. These are the so-called damped modes, which can be shown to satisfy a different form of orthogonality [6]. In a similar manner, the response of a continuous member with non-proportional viscous damping was determined by O'Kelley [8] by using the damped mode shapes. In addition, Caughey and O'Kelley [8] studied the limitations of modal analysis for this kind of general viscous damping by determining necessary and sufficient conditions for which the undamped modes can be used to uncouple the dynamic motion (i.e. proportional viscous damping). However, few applications have appeared in the literature applying damped modal analysis to continuous members with viscous damping. For example, Lund [9] used damped eigenfunctions to find the dynamic response of a rotating shaft on damped bearings. The approach taken was not general and it involved a great deal of tedious algebraic manipulations.

Thus far, very little work has been done to generalize any of the above results to any damped structural member with arbitrary non-homogeneous boundary conditions, in-span conditions, and non-self-adjoint equations. This is true even for viscous damping which is the most popular model because of its mathematical convenience rather than its

physical exactness. O'Kelley [7] did use a general approach to resolve the dynamic response of viscously damped continuous members, but his remarks were restricted to self-adjoint systems with homogeneous boundary conditions and no in-span conditions. Meirovitch [2] also considered self-adjoint systems of equations for both viscous and structural damping with only homogeneous boundary conditions and no in-span conditions. The most general formulation was presented by Pfennigwerth [10] and later extended by Cinelli and Pilkey [11]. However, only undamped continuous structural members were considered.

In this work, a comprehensive theory is presented for the dynamic response of continuous damped structural members. A general set of formulas is derived that explicitly provides the ingredients necessary to form the modal solution due to arbitrary loadings. These ingredients include the necessary normal modes (shown to possess the proper orthogonality) and the time-dependent uncoupled coordinates. It is shown that a very general type of problem can be resolved with this approach. That is, linear structural members can be self-adjoint or non-self-adjoint, possess homogeneous or non-homogeneous boundary conditions, or have in-span conditions such as in-span supports. In addition, members with two or three independent spatial variables may be treated as easily as those with one spatial dimension. Knowing only the differential equations of the structural member, the dynamic response is explicitly written out, thus avoiding the tedious algebraic manipulations which are now required when solving for a particular member. Finally, this general approach also exposes the mathematical limitations imposed by the physical model which prohibit the uncoupling of some damped structural members.

General Formulation for the Dynamic Response of Continuous Structural Members with Viscous Damping

The starting point of this general formulation is the governing differential equation of motion of the structure. The class of viscously damped structural members to be considered may be defined by the following equation of motion:

$$Du(x,t) = \sum_{j=1}^{2} A_j(x)\partial_j u(x,t) - F(x,t) \tag{1a}$$

with initial conditions:

$$u(x,0)=u_0(x) \quad ; \quad \partial u(x,0) = \dot{u}_0(x) \tag{1b}$$

and the time-dependent boundary and in-span conditions,

$$L_i u(x,t) = P_i(x,t) \quad \text{on S, the body surface and in-span locations} \tag{1c}$$

where,

$F(x,t)$ = N-dimensional column vector of body forces

$D(x)$, $L_i(x)$ = N-square spatial matrix linear differential operators

$A_j(x)$ = N-square spatial matrix

$P_i(x,t)$ = N-dimensional column vector of non-homogeneous boundary and in-span conditions

$u(x,t)$ = N-dimensional column vector of dependent state variables

$u_0(x), \dot{u}_0(x)$ = N-dimensional vectors of prescribed state variable initial conditions

$\partial_j = \partial^{(j)}/\partial t^{(j)}$

and x represents the independent spatial coordinates x_1, x_2, and x_3.

Since equations (1) are matrix equations, they represent N distinct governing differential equations of motion and their associated boundary, in-span, and initial conditions. Note that lumping the boundary and in-span conditions into one equation (equation (1c)) is reasonable because boundary terms may really be considered in-span conditions at the boundary of the member. Furthermore, the case of homogeneous boundary and in-span conditions may be considered to be a special case of this development wherein $P_i(x,t)$ is set equal to zero.

In order to determine the transient response using a modal analysis, normal modes of the structural member must be known. Depending on whether the member possesses non-proportional, proportional, or no damping, the normal modes used in the modal expansion will be different. It has been shown by Caughey and O'Kelley [8] that damped continuous linear systems can be uncoupled by either undamped or damped modes and these constitute two mutually exclusive classes of problems. Systems which require only undamped modes for uncoupling are called classical and all others are non-classical. Necessary and sufficient conditions have also been formulated to determine when a viscously damped self-adjoint linear continuous system is classical or non-classical [8].

Solution of a so-called free vibration problem will yield the normal modes and corresponding eigenvalues necessary for modal analysis. Classical normal modes will result if damping is excluded from the free vibration problem; non-classical modes if damping is included. By definition, the damped or undamped free vibration problem represents motion in which all external forces and prescribed deflections have been set equal to zero. When the prescribed conditions occur on the boundary,

they are called non-homogeneous boundary conditions. Otherwise, they constitute in-span conditions. A member undergoing free motion will vibrate due to forces inherent in it, having been set in motion by prescribed initial conditions.

For an undamped member, it can be shown that the free motion response has a separable form of solution given by

$$u(x,t) = \psi_n(x)e^{i\lambda_n t} \quad . \tag{2}$$

where λ_n is the natural frequency and $\psi_n(x)$ is the vector of undamped (or classical) mode shapes. The undamped governing equation of motion is given by setting the damping matrix $A_1 = 0$ in equation (1a). By substituting the solution given by equation (2) into the governing equation, the undamped free vibration problem is obtained

$$D\psi_n(x) = -(\lambda_n)^2 A_2(x)\psi_n(x) \tag{3a}$$

with the boundary and in-span conditions

$$L_i\psi_n(x) = 0 \qquad \text{on } S \tag{3b}$$

It will be shown that only the classical modes $\psi_n(x)$ are needed to express a modal series solution of equation (1) for undamped or proportionally damped systems.

Guided by equation (2), when a member has non-proportional viscous damping the damped free motion solution may be assumed to have the form

$$u(x,t) = v_n(x)e^{i\omega_n t}e^{-\gamma_n t} \tag{4}$$

where $e^{-\gamma_n t}$ represents the temporal decay due to damping in the structural member. By defining the complex frequency to be

$$S_n = -\gamma_n + i\omega_n \tag{5}$$

equation (4) may be written as

$$u(x,t) = v_n(x)e^{S_n t} \tag{6}$$

in analogy with equation (2). Now S_n and $v_n(x)$ represent the eigenvalue and vector of mode shapes, respectively, corresponding to the damped structural member. Note that in contrast to the undamped member, the eigenvalues and mode shapes are complex-valued for underdamped structural members.

As was done for the undamped case, the damped free vibration problem may now be derived by substituting equation (6) into equation (1a) and setting all applied loadings equal to zero. This results in the equation

$$Dv_n(x) = \sum_{j=1}^{2} (S_n)^j \; A_j(x)v_n(x) \tag{7a}$$

with the boundary and in-span conditions

$$L_i v_n(x) = 0 \qquad \text{on } S \tag{7b}$$

Equations (3) and (7) constitute eigenvalue problems. The undamped free vibration problem led to equation (3), which for self-adjoint systems is a higher-order generalization of the traditional Sturm-Liouville boundary value problem. When damping is included in the free

vibration problem, a non-classical eigenvalue problem results in which the eigenvalue appears non-linearly. Regardless of whether the system is classical or non-classical, it will be assumed that the solution consists of a denumerably infinite set of eigenvalues and eigenfunctions, which are complete. Also since repeated eigenvalues are of limited practical use, only zero multiplicity will be considered in this work. Hence, the corresponding eigenfunctions will be assumed to be linearly independent.

The previous undamped and damped free vibration problems may be combined into the single general form:

$$Du_n(x) = \sum_{j=\alpha}^{2} (-1)^{(\alpha-1)} (\mu_n)^j \; A_j(x) u_n(x) \tag{8a}$$

with the boundary and in-span conditions

$$L_i u_n(x) = 0 \qquad \text{on } S \tag{8b}$$

Here u_n and μ_n are the eigenfunction and eigenvalue which can be classical or non-classical, depending on whether damping is included in the free motion problem. That is, assign the following significance:

$$(u_n, \mu_n) = \begin{cases} (\Psi_n, \lambda_n), & \text{Undamped Free Motion} \\ (v_n, S_n), & \text{Damped Free Motion} \end{cases} \tag{9}$$

Also, for no damping set $\alpha = 2$ and when damping is present assign $\alpha = 1$. The remaining discussion in this work will use this notation whenever possible to enhance the general approach underlying this work.

Associated with both the undamped and damped eigenvalue problems, an adjoint eigenvalue problem may be described. Using the general notation in equation (9), the adjoint problem takes the form

$$\tilde{D}\,\tilde{u}_n(x) = \sum_{j=\alpha}^{2} (-1)^{(\alpha-1)} (\mu_n)^j \tilde{A}_j(x)\,\tilde{u}_n(x) \tag{10a}$$

with homogeneous adjoint boundary and in-span conditions

$$\tilde{L}_i \tilde{u}_n(x) = 0 \qquad \text{on } S \tag{10b}$$

where

$\tilde{D}(x)$, $\tilde{L}_i(x)$ = N-square algebraic adjoint spatial matrix linear differential operators

$\tilde{A}_j(x)$ = N-square algebraic adjoint spatial matrix

$\tilde{u}_n(x)$ = N-dimensional column vector of algebraic adjoint eigenfunctions corresponding to the nth eigenvalue, either classical or non-classical (see equation (9)).

The eigenvalue of the original free vibration problem is also the eigenvalue of the adjoint problem, for both undamped and damped cases (see References (12) and (13)). Note that in equation (10a) the matrix $\tilde{A}_j(x)$ has been restricted not to be a differential operator; therefore, the algebraic adjoint $\tilde{A}_j(x)$ is equivalent to the transpose of $A_j(x)$.

The adjoint operator and the adjoint boundary and in-span conditions denoted by $\tilde{D}(x)$ and $\tilde{L}_i(x)$ in equation (10a) are not known a priori from the original statement of the eigenvalue problem. However, these operators are formally shown to exist by the use of various forms of the Green's identity. One form which is valid for homogeneous boundary conditions and no in-span conditions is given by

$$<\tilde{u}_n, D\, u_m> - <u_m, \tilde{D}\, \tilde{u}_n> = 0 \tag{11}$$

where the notation $<\tilde{u}_n, D\, u_m>$ denotes the inner product,

$$<\tilde{u}_n, D\, u_m> = \int_{domain} \tilde{u}_n \cdot (D\, u_m)dx \tag{12}$$

and u_m and $\tilde{u}_n$ represent either the classical or non-classical vector of mode shapes and the associated vector of adjoint modes, respectively. When in-span conditions are present, it can be shown [13] that the Green's identity takes the form,

$$<\tilde{u}_n, D\, u_m> = <u_m, \tilde{D}\, \tilde{u}_n> + B(\tilde{u}_n, u_m) \tag{13}$$

where $B(\tilde{u}_n, u_m)$ is a bilinear function of the state variable vectors u_m and $\tilde{u}_n$ which represents the boundary term and in-span condition at each location. Note that the eigenfunctions u_m and $\tilde{u}_n$ must be differentiable to the extent demanded by the operators $\tilde{D}$ and D when no in-span conditions are present. Otherwise, these functions need only be differentiable at all locations not coinciding with the in-span conditions.

Using the proper form of the Green's identity, it may be used to identify the adjoint free vibration problem for any classical or non-classical structural member. Briefly, the left-hand-side of equation (13) is formed, and then integrated by parts so that the operator $\tilde{D}$ and bilinear functional B are identified for any member. Several practical illustrations of deriving equations (10) are provided in Reference [13]. Note that in general, the solution of the adjoint problem given symbolically by equation (10) will be different from that of the original free motion

in equation (8). However, when the problem is self-adjoint these two solutions will be identical. The advantage of this is obvious: only one eigenvalue problem must be solved. Moreover, an orthogonality relation which involves only the original eigenfunctions is needed to uncouple the transient response.

The modal analysis solution of a dynamic vibration problem is characterized by the form

$$u(x,t) = \sum_{m=1}^{\infty} q_m(t) u_m(x) \tag{14}$$

where u(x,t) is the vector of state variables describing the deflection and internal forces present in the member at any given location and time. The normal modes or eigenfunctions $u_m(x)$ will be assumed to be known. In the classical case, $u_m(x)$ can often be determined analytically in closed form for many simple members [2]. However, when non-proportional damping is present in a structural member an analytical expression for the eigenfunctions cannot be found and instead a numerical technique must be used. When the structural member has one spatial direction, transfer matrices are particularly well-suited to providing an accurate means of solution. A description of this technique for numerically evaluating the eigenfunctions will be given in a later section.

In this work, it is assumed that the modal series solution given by equation (14) is convergent for all physically admissible problems. Furthermore, it will be assumed that the series is uniformly convergent except at locations where discontinuities exist such as in-span supports. At these locations, the series may or may not converge. Hence, it may

be concluded that the dynamic response will be determined if the proper eigenfunctions are shown to be theoretically orthogonal, and the so-called normal coordinates $q_m(t)$ can be found via an uncoupled differential equation for each m-coordinate.

To find the dynamic solution of a general damped structural member using modal analysis, an orthogonality relation must be shown to exist between the eigenfunctions. Because both the classical and non-classical free vibration problems can be non-self-adjoint, orthogonality will be shown to prevail between the vector of original eigenfunctions $u_m(x)$ and the associated vector of adjoint functions $\tilde{u}_m(x)$. This is commonly referred to as a biorthogonality relation. When the free motion problem is self-adjoint, the biorthogonal form reduces to an orthogonality condition between the original eigenfunctions only. In this work both forms will be referred to as an orthogonality relation.

To derive the general orthogonality relation, begin with the form of Green's identity given in equation (11). Note that even if the member possesses in-span conditions, the extended Green's identity in equation (13) reduces to the homogeneous form by $B(\tilde{u}_n, u_m) = 0$. Substitute equations (8a) and (10a) into equation (11) to give

$$\sum_{j=\alpha}^{2} (-1)^{(\alpha-1)} [(\mu_m)^j < \tilde{u}_n, A_j u_m > - (\mu_n)^j < u_m, \tilde{A}_j \tilde{u}_n >] = 0 \qquad (15)$$

Another form of the Green's identity may be given by

$$<u_m, \tilde{A}_j \tilde{u}_n> \quad <\tilde{u}_n, A_j u_m> \qquad j = 1,2 \qquad (16)$$

which is valid for A_j, a matrix of scalar spatial elements. A common inner product may now be factored out of equation (15) by applying equation (16), so that equation (15) becomes

$$\sum_{j=\alpha}^{2} (-1)^{(\alpha-1)}[(\mu_m)^j - (\mu_n)^j] <\tilde{u}_n, A_j u_m> = 0 \tag{17}$$

The form of the orthogonal relation depends on whether the system is classical or non-classical. Considering the former case, let $\alpha = 2$ and $u_n(x)$ and μ_n take on the values given by equation (9). Then equation (17) may be rewritten as

$$(\beta_n - \beta_m) <\tilde{\psi}_n, A_2\psi_m> = 0 \tag{18}$$

where $\beta_n = \lambda_n^2$. When $\beta_n \neq \beta_m$ the inner product must be zero to satisfy equation (18); otherwise, it does not vanish. Thus, the orthogonal relation for classical systems may be expressed in the form:

$$<\tilde{\psi}_n, A_2\psi_m> = \delta_{mn} Q_n \tag{19}$$

where Q_n represents the classical norm given by

$$Q_n = <\tilde{\psi}_n, A_2\psi_n> \tag{20}$$

and δ_{mn} is the Kronecker delta. Note that in the unlikely event that the norm is zero, the vector of adjoint eigenfunctions $\tilde{\psi}_n$ may be replaced by its complex conjugate. The functions $\tilde{\psi}_n$ and ψ_m would then be orthogonal in the so-called Hermitian sense.

For non-classical systems, $\alpha = 1$ and u_n and μ_n become v_n and S_n, the vector of damped eigenfunctions and correspnding complex

eigenvalues in equation (15). After applying the same argument that led to equation (19), non-classical orthogonality may be expressed by

$$(S_m - S_n)\ [(S_m + S_n)\ \langle \tilde{v}_n, A_2 v_m \rangle + \langle \tilde{v}_n, A_1 v_m \rangle\ ] = \delta_{mn} N_n \tag{21}$$

where now N_n is the non-classical norm given by

$$N_n = 2S_n \langle \tilde{v}_n, A_2 v_n \rangle + \langle \tilde{v}_n, A_1 v_n \rangle \tag{22}$$

Note that A_1 and A_2 may contain complex elements and therefore, both norms, N_n and Q_n, may be complex-valued.

To complete the modal solution as given by equation (14) the uncoupled coordinates $q_m(t)$ must be determined. This is accomplished by transforming the governing equation of motion into an uncoupled differential equation. Uncoupled here means that a different equation exists for each m-eigenvalue, independent of all other values. To effect this transformation, begin with the Green's identity in the general extended form

$$\langle \tilde{u}_m, Du \rangle - \langle u, \tilde{D}\, \tilde{u}_m \rangle = B(\tilde{u}_m, u) \tag{23}$$

This equation is valid for a member with non-homogeneous boundary conditions and in-span conditions ($B \neq 0$) as well as for homogeneous boundary and in-span conditions ($B = 0$). Substitute equation (1a) and (10a) into the above equation to obtain

$$\sum_{j=1}^{2} \partial_j \langle \tilde{u}_m, A_j u \rangle - \sum_{j=\alpha}^{2} (-1)^{(\alpha-1)} (\mu_m)^j \langle u, \tilde{A}_j \tilde{u}_m \rangle = \langle \tilde{u}_m, F \rangle + B(\tilde{u}_m, u) \tag{24}$$

The bilinear form, once identified for a particular example, will retain its form. By applying the following form of the Green's identity,

$$<u,\tilde{A}_j\tilde{u}_m> = <\tilde{u}_m,A_ju> \qquad j = 1,2 \tag{25}$$

equation (24) may be rewritten as

$$\sum_{j=1}^{2} \partial_j <\tilde{u}_m A_j u> - \sum_{\alpha}^{2} (-1)^{(\alpha-1)} (\mu_m)^j <\tilde{u}_m,A_ju>$$
$$= <\tilde{u}_m,F> + B(\tilde{u}_m,u) \tag{26}$$

To reduce this equation further, the value of α must be known. For a classical system, $\alpha = 2$ and $\tilde{u}_m$ and μ_m are replaced by the adjoint classical mode shape $\tilde{\psi}_m$ and the corresponding eigenvalue λ_m. Noting these changes and expanding the sum, equation (26) becomes

$$\partial_2 + (\lambda_m)^2] <\tilde{\psi}_m,A_2u> + \partial<\tilde{\psi}_m,A_1u>$$
$$= <\tilde{\psi}_m,F> + B(\tilde{\psi}_m,u) \tag{27}$$

A common inner product may not be factored out due to the presence of the damping term A_1. At this point the proportionality assumption must be invoked,

$$A_1u = aA_2u - b\ D\ u \tag{28}$$

where a and b are constants of proportionality. That is, the damping terms (A_1) are assumed to be proportional to the mass (A_2) and/or

stiffness (D) of the member. Another form of this condition may be found by multiplying equation (28) by $\tilde{\psi}_m$ and integrating over the domain to obtain

$$<\tilde{\psi}_m, A_1 u> = a <\tilde{\psi}_m, A_2 u> - b <\tilde{\psi}_m, Du> \tag{29}$$

Apply the extended Green's identity to the last term on the right-hand side and then substitute the classical form of equation (10a) into equation (29). After some rearrangement obtain the equation

$$\begin{aligned} <\tilde{\psi}_m, A_1 u> &= (a+b\lambda_m^2) <\tilde{\psi}_m, A_2 u> \\ &\quad - b\, B(\tilde{\psi}_m, u) \end{aligned} \tag{30}$$

With this form of the proportionality assumption, return to equation (27). After substituting equation (30), obtain the equation

$$\begin{aligned} \ddot{\xi}_m(t) + (a+b\lambda_m^2)\, \dot{\xi}_m + \lambda_m^2 \xi_m \\ = <\tilde{\psi}_m, F> + (1+b\partial) B(\tilde{\psi}_m, u) \\ = F_m^{\dagger}(t) \end{aligned} \tag{31}$$

where $(\dot{\ }) = \partial(\ )/\partial t$ and where ξ_m has been assigned the value

$$\xi_m(t) = <\tilde{\psi}_m, A_2 u> \tag{32}$$

Note that ξ_m is a function of time alone because the spatial variables have been eliminated by the definite integral. The solution of equation (31) may be easily found to be

$$\xi_m(t) = e^{-\lambda_m \zeta_m t}[\cos \alpha_m t + \frac{\lambda_m \zeta_m}{m} \sin \alpha_m t]\ \xi_m(0)$$

$$+ e^{-\lambda_m \zeta_m t}[\frac{\sin \alpha_m t}{\alpha_m}]\ \dot{\xi}_m(0) \tag{33a}$$

$$+ \int_0^t F_m^{\dagger}(\tau)\ e^{-\lambda_m \zeta_m (t-\tau)}\ \frac{\sin \alpha_m (t-\tau)}{\alpha_m}\ d\tau$$

where

$$\zeta_m = \frac{1}{2}\left(\frac{a}{\lambda_m} + b\ \lambda_m\right) \tag{33b}$$

and

$$\alpha_m = \lambda_m \sqrt{1 - \zeta_m^2} \tag{33c}$$

where the initial conditions, $\dot{\xi}_m(0)$ are given by equation (32) by replacing u by u_0 and $\dot{u}_0$, respectively. Note that as a special case of a classical system, equation (31) may be used to represent an undamped member. For this case, set a = b = 0 in equations (31) and (33) and continue to use equation (32).

In an identical manner, an expression for the normal coordinates for a non-classical system can be derived. Beginning with the general equation (26), let $\alpha = 1$ and $\tilde{u}_m$ and μ_m then become the adjoint non-classical mode shape and the complex eigenvalue, respectively. When these changes are introduced into equation (26) and the sums are expanded, it may be expressed as

$$(\partial - S_m)\ [\ <\tilde{v}_m, A_1 u> + (\partial + S_m)\ <v_m, A_2 u>\ ]$$

$$= <\tilde{v}_m, F> + B(\tilde{v}_m, u) \tag{34}$$

The term in square brackets is a function of time alone so that it may be represented by the variable

$$\eta_m(t) = <\tilde{v}_m, A_1 u> + <\tilde{v}_m, A_2 \partial v> + S_m\ <\tilde{v}_m, A_2 v> \tag{35}$$

Using the above definition, equation (34) may be written as

$$\dot{\eta}_m(t) - S_m \eta_m = <\tilde{v}_m, F> + B(\tilde{v}_m, u) \tag{36}$$

This equation is essentially an ordinary, first order, uncoupled differential equation which has the solution

$$\eta_m(t) = e^{S_m t}\, \eta_m(0) +$$

$$\int_0^t e^{S_m(t-\tau)}\ [\ <\tilde{v}_m, F\ + B(\tilde{v}_m, u)]\ d\tau \tag{37}$$

and $\eta_n(0)$ is found by evaluating equation (35) at $t = 0$.

The solutions given by equations (33) and (37) represent the first step in finding the uncoupling coordinates $q_m(t)$ necessary to complete the modal solution given by equation (14). To avoid confusion, let $q_m(t)$ assume the values

$$q_m(t) = \begin{cases} f_m(t) & , \quad \text{Classical Systems} \\ g_m(t) & , \quad \text{Non-Classical Systems} \end{cases} \quad . \tag{38}$$

in equation (14). Considering first the non-classical systems, recall that the ultimate goal is to find $g_m(t)$ in the series solution,

$$u(x,t) = \sum_{m=1}^{\infty} g_m(t)v_m(x) \tag{39}$$

Since both η_m and g_m are functions of time alone, a relationship between them is sought. For no in-span conditions equation (39) is assumed to be uniformly convergent. When in-span conditions are present, discontinuities of some of the state variables will occur at these locations. However, at all other points along the member, the modal expansion may be assumed to remain uniformly convergent. That is, the state variables will be assumed to be continuous throughout all subintervals which are punctuated by these discontinuities. Such functions are often referred to as sectionally continuous. For any well-defined physical problem, this modal expansion may converge very slowly or even diverge at these points of discontinuity. However, the analyst is frequently not interested in the response at precisely these points so that the formulation in equation (39) is still useful as long as these locations are excluded from the analysis. Then, at all points for which the series is uniformly convergent, it may be differentiated with respect to time. Guided by equation (37), $g_m(t)$ will have an exponential form so it follows that

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} S_m g_m(t)v_m(x) \tag{40}$$

Substitute equations (39) and (40) into the definition of $\eta_n(t)$, then by assuming that the integral of an infinite series is equivalent

to an infinite series of the integrals, this equation may be rewritten as

$$\eta_n(t) = \sum_{m=1}^{\infty} g_m \, [\langle\tilde{v}_n, A_1 v_m\rangle + S_m \langle\tilde{v}_n, A_2 v_m\rangle + S_n \langle\tilde{v}_n, A_2 v_m\rangle] \tag{41}$$

Using non-classical orthogonality and the associated norm in equations (21) and (22), equation (41) becomes

$$g_m(t) = \frac{\eta_m(t)}{N_m} \tag{42}$$

Since $\eta_m(t)$ is known from equation (37), $g_m(t)$ is also known. Hence, the dynamic response may be written out as shown in equation (39).

For the classical system, the same procedure may be followed to give

$$f_m(t) = \frac{\xi_m(t)}{Q_m} \tag{43}$$

a well-known result for undamped structural members. The modal solution then takes the form

$$u(x,t) = \sum_{m=1}^{\infty} \frac{\xi_m(t)}{Q_m} \psi_m(x) \tag{44}$$

In summary, the dynamic response of any structural member with equations of motion given by equation (1) may be found by applying either of the following two series:

$$u(x,t) = \begin{cases} \sum_{m=1}^{\infty} \dfrac{\xi_m(t)}{Q_m} \psi_m(x) \quad , \quad \text{Classical} & (45a) \\ \sum_{m=1}^{\infty} \dfrac{\eta_m(t)}{N_m} v_m(x) \quad , \quad \text{Non-Classical} & (45b) \end{cases}$$

where $\xi_m(t)$ and $\eta_m(t)$ are given by equations (33) and (37), respectively.

Acceleration Method of the Dynamic Response of Continuous Structural Members with Viscous Damping

In the previous section, the dynamic response of a completely general viscously damped structural member was formulated as an infinite series solution. This form of the solution is often referred to as the displacement method. The advantage of this approach lies in that it will usually converge to the response with only a small number of terms. If the series solution requires many terms, it is often possible to overcome the slow convergence by modifying the displacement solution. One technique which may be used to achieve this goal is called the acceleration method. In this section, this form of the modal solution will be derived for a structural member with proportional, non-proportional, or no viscous damping.

Acceleration Method for Proportionally Damped Structural Members

The structural member treated in this subsection possesses the so-called proportional damping, which satisfies the condition given in equation (30). When a = b = 0 the member is undamped, and so undamped members may be considered to be a special case of proportional damping.

As mentioned in the previous section, proportionally damped structural members are categorized as classical systems. Hence, only the undamped modes are required to uncouple the dynamic response which is given by equation (44). Recall that the uncoupling coordinates $\xi_m(t)$ to obtain

$$\xi_m(t) = -\frac{1}{\lambda_m^2}\,[\ddot{\xi}_m + (a+b\lambda_m)\,\dot{\xi}_m - F_m^{\dagger}] \tag{46}$$

Substitute this equation into equation (44) and arrive at the expression

$$u(x,t) = \sum_{m=1}^{\infty} \frac{\psi_m F_m^{\dagger}}{\lambda_m^2 Q_m} - \sum_{m=1}^{\infty} \frac{\psi_m}{\lambda_m^2 Q_m}\,[\ddot{\xi}_m + (a+\lambda_m^2 b)\,\dot{\xi}_m] \tag{47}$$

The first term on the right-hand side of this equation is a function of the spatial and temporal variables. However, time only enters through the generalized forcing function $F_m^{\dagger}$ which represents applied loadings and non-homogeneous boundary and in-span conditions. Because these are prescribed, $F_m^{\dagger}$ may be determined at every instant of time for which this first term may be considered to be a function of x alone. With this understanding, define this expression as a static or quasi-static term, given by

$$v_s^{\dagger} = \sum_{m=1}^{\infty} \frac{\psi_m F_m^{\dagger}}{\lambda_m^2 Q_m} \tag{48}$$

For additional convenience, define

$$C_m^{\dagger}(t) = -\frac{1}{\lambda_m^2}\,[\ddot{\xi}_m + (a+b\lambda_m^2)\,\dot{\xi}_m] \tag{49}$$

so that Equation (47) may be rewritten as

$$u(x,t) = v_s^\dagger + \sum_{m=1}^{\infty} \frac{C_m^\dagger(t)}{Q_m} \psi_m(x) \tag{50}$$

Note that $u(x,t)$, $v_s^\dagger$, and $\psi_m(x)$ are column vectors representing the dynamic static, and classical mode state variables of the member, respectively.

The underlying goal of the acceleration method is to extract a static solution from the modal expansion. In effect, the acceleration solution achieves a "jump" on the dynamic response by beginning with the static solution. To define the terms in equation (50) in greater detail, a specific solution form is needed. Recall the solution for $\xi_m(t)$ given by equation (33). To extract the static solution, integrate the last term in equation (33a) by parts twice. Substitute the resulting equation into equation (45a), and then compare this expression with equation (50) for determining $u(x,t)$. In this way, the alternative expression for $C_m^\dagger(t)$ may be identified as

$$C_m^\dagger(t) = \frac{\zeta_m^2 F_m^\dagger(t)}{(1-\zeta_m^2)\lambda_m^2} + e^{-\lambda_m \zeta_m t} [\cos \alpha_m t + \frac{\zeta_m \lambda_m}{\alpha_m} \sin \alpha_m t] [\xi_m(0) - \frac{F_m^\dagger(0)}{\alpha_m^2}]$$

$$+ e^{-\lambda_m \zeta_m t} \frac{\sin \alpha_m t}{\alpha_m} [\dot{\xi}_m(0) - \frac{1}{\alpha_m^2} \frac{dF_m^\dagger(0)}{dt}] + \tag{51}$$

$$- \frac{1}{\alpha_m^2} \int_0^t \{[\frac{d}{d\tau^2} + (a + b\lambda_m^2) \frac{d}{d\tau} + (\lambda_m \zeta_m)^2] F_m^\dagger(\tau)$$

$$e^{-\lambda_m \zeta_m (t-\tau)} \frac{\sin \alpha_m (t-\tau)}{\alpha_m} \, d\tau$$

To summarize, an alternative solution of the transient response problem has been derived in equation (50). The static solution, $v_s^\dagger$ is obtained from equation (48) and the temporal coefficient $C_m^\dagger(t)$ is given in equation (51). Note that $v_s^\dagger$ may also be found by solving equations (1) in which $u = v_s^\dagger$ and $A_1 = A_2 = 0$ by any suitable method.

As special cases of the above acceleration solution, note that if the member was undamped merely setting ζ_m equal to zero ($a = b = 0$) and $\alpha_m = \lambda_m$ would allow uncoupling of the response. In addition, the case of the homogeneous boundary conditions and no in-span conditions would affect only the value of the generalized force term defined in equation (31). That is, set $B(\tilde{\psi}_m, u)$ equal to zero.

Acceleration Method for Non-Proportionally Damped Structural Members

The assumption of proportional damping is relaxed in this subsection. Hence, the dynamic response of such structural members will be uncoupled with the non-classical or damped mode shapes. Recall that for general viscous damping the uncoupling coordinates $\eta_m(t)$ are found by solving equation (36). To find the acceleration method for this case, begin by solving this equation for $\eta_m(t)$:

$$\eta_m(t) = \frac{1}{S_m} (\dot{\eta}_n - F_m) \tag{52}$$

Where the non-classical form of the generalized forcing function,

$$F_m(t) = < \tilde{v}_m, F > + B(\tilde{v}_m, u) \tag{53}$$

has been employed. Upon substitution of equation (52) into equation

(45b), obtain the expression for the dynamic response

$$u(x,t) = -\sum_{m=1}^{\infty} \frac{v_m F_m}{S_m N_m} + \sum_{m=1}^{\infty} \frac{\dot{\eta}_m}{S_m N_m} v_m \tag{54}$$

In identical fashion to the previous subsection, the first series in this equation may be recognized as being quasi-static so that it may be rewritten in the form

$$u(x,t) = v_s + \sum_{m=1}^{\infty} \frac{C_m(t)}{N_m} v_m(x) \tag{55}$$

where v_s is the static solution for the non-classical system.

Following the pattern of the previous subsection, the solution for the temporal coefficient $\eta_m(t)$ in equation (37) is integrated by parts twice to extract the static portion of the response. Then the resulting equation for $\eta_m(t)$ is substituted back into the displacement modal expansion in equation (45b). This equation is then compared with equation (55), the static portion of the solution is identified, and the coefficient $C_m(t)$ is given by

$$C_m(t) = -\frac{1}{S_m^2} \frac{dF_m(t)}{dt} + e^{S_m t} \left[\frac{1}{S_m^2} \frac{dF_m(0)}{dt} + \frac{F_m(0)}{S_m} + \eta_m(0) \right] + \int_0^t \frac{e^{S_m(t-\tau)}}{S_m^2} \frac{d^2 F_m}{d\tau^2} d\tau \tag{56}$$

Hence, the acceleration form of the modal solution is given by equation (55) where $C_m(t)$ is found from the above equation. No simplifying assumptions have been made in this derivation, so that the structural member can possess general (non-proportional) viscous properties, have time-dependent non-homogeneous boundary and in-span conditions, and accept any arbitrary loadings. For a structural member for which the usual modal expansion is prohibitive because of the many terms needed for adequate convergence, this acceleration form of the solution may still provide a viable means of finding the dynamic response using modal analysis.

Dynamic Response of a Viscoelastic Structural Member on a Proportional Viscous Foundation: Part I

In this section the response of a general viscoelastic structural member will be found using modal analysis. The constitutive relation to be used will be of the hereditary integral variety. However, it can be shown that the final solution may be altered to allow a differential operator form of the material law to also be used (see Reference [13]). As mentioned previously, Valanis [3] was the first to resolve the solution of such a general "viscoelasto-kinetic" problem into a superposition of a viscoelastic quasi-static and a dynamic elastic solution. This section will go beyond the work of Valanis and others (Robertson, [4],[5] etc.) by considering a more general approach applicable to both self-adjoint and non-self-adjoint systems, which may also possess a proportionally damped viscous foundation (hereafter a viscoelastic member on a viscous foundation will be referred to as a

damped viscoelastic structural member)*. In addition, non-homogeneous surface tractions or internal forces, such as shear forces or moments, may be included in-span or on the boundaries of the member. However, prescribed displacements on the surface are restricted to be homogeneous. This limitation has also been used by Valanis [3]. It will be shown in this section that this restriction is necessary to uncouple the dynamic response of viscoelastic structural members using the general approach involving the Green's identity.

The starting point of this general formulation will again be the differential equations governing the motion of the member given by

$$G*Du(x,t) = \sum_{j=1}^{2} A_m(x)\ \partial_j u(x,t) - F(x,t) \tag{57a}$$

with the initial conditions,

$$u(x,0) = u_0(x); \qquad \partial u(x,0) = \dot{u}_0(x) \tag{57b}$$

and the time-dependent boundary and in-span conditions,

$$G*L_i u(x,t) = P_i(x,t) \qquad \text{on } S \tag{57c}$$

where $P_i(x,t)$ is a vector containing the prescribed non-homogeneous surface tractions and homogeneous prescribed displacements. The notation used for viscous damping remains unchanged for the general viscoelastic problem. Hence, proportional viscous damping, such as found in a foundation, is included in the matrix $A_1(x)$. In addition,

*In fact, it can be shown [13] if the viscous foundation is non-proportional, a modal solution is not possible.

equation (57c) contains all boundary conditions including the non-homogeneous displacements. Although the prescribed displacements do not involve G*, equation (57c) can be used to represent these boundary conditions because they are homogeneous and G is a monotonically decreasing function of time. Note that equations (57) apply only to a linear isotropic viscoelastic material, deforming under isothermal conditions with a constant Poisson's Ratio, and which is incompressible. Nevertheless, this general form of the equations of motion describe a wide class of viscoelastic members on a proportionally damped viscous foundation. An example of such a structural member will be shown to fit this general theory in the Applications section.

Recall that the solution of a general self-adjoint viscoelastic member has been given by Valanis to be a superposition of a quasi-static viscoelastic and a dynamic elastic solution. Guided by this observation and the fact that the viscous damping is restricted to be proportional, it is presumed that the undamped mode shapes may be used to uncouple the response of any complicated member. Recall from a previous section that the classical eigenfunctions were found by considering the undamped free vibration problem given by equations (3). In addition, the viscoelastic member may be non-self-adjoint, so that a biorthogonality relation will be needed in deriving the elastic modal expansion. Hence, use will be made of the adjoint undamped free motion problem given by equations (10) along with equation (9). Because the elastic undamped free vibration problems are identical to the proportional viscous damping case, the orthogonality relation given by equations (19) and (20 is also unchanged.

Assume that the solution of the general viscoelastic dynamic problem may be expressed in the usual modal series form,

$$u(x,t) = \sum_{m=1}^{\infty} h_m(t)\, \tilde{\psi}_m(x) \tag{58}$$

The classical eigenfunctions, $\psi_m(x)$, can be easily found for a variety of structural members. Thus, only the temporal coefficients $h_m(t)$ need to be determined to complete the modal solution. In order to derive an uncoupled equation for these coefficients, proceed as follows. The Green's identity, in extended form, may be written as

$$< \tilde{\psi}_m, D(G{*}u) > - < (G{*}u), \tilde{\tilde{D}}\psi_m > = B(\tilde{\psi}_m, (G{*}u)) \tag{59}$$

The term, $(G{*}u)$, is really a function of both the spatial and temporal variables and its use in equation (59) is valid if it is differentiable to the extent demanded by the operator D. This is assumed to be the case. The bilinear form $B(\tilde{\psi}_m, (G{*}u))$ is determined in exactly the same way as for the elastic member. That is, the inner product $< \tilde{\psi}_m, D(G{*}u) >$ is formed and then integrated by parts with respect to the spatial coordinates. Note that the appearance of G does not effect the integration by parts because G is a function of time. The boundary and in-span conditions are then grouped to form the bilinear functional $B(\tilde{\psi}_m, (G{*}u))$. Since the prescribed displacements on the boundary are zero, this bilinear form contains only the non-homogeneous surface tractions which involve the relaxation modulus G(t). In fact, it is this form of $B(\tilde{\psi}_m, (G{*}u))$ that prohibits non-homogeneous displacements from being prescribed on the boundaries because these may not be expressed in the form $G{*}u$.

Before returning to equation (59), note that a property of the convolution is that

$$G*(Du(x,t)) = (Du(x,t))*G \tag{60}$$

where Du(x,t) may be treated as an arbitrary function of x and t. Since D is a spatial differential operator it does not effect G, a function of time, so that

$$G*Du = Du*G = D(u*G) = D(G*u) \tag{61}$$

With this equation, the governing equation of motion, equation (57a) becomes

$$D(G*u = \sum_{j=1}^{2} A_j(x)\ \partial_j u(x,t) - F(x,t) \tag{62}$$

Now substitute equations (10a) and (62) into the Green's identity in equation (59) to obtain

$$\begin{aligned}\partial < \tilde{\psi}_m, A_1 u > + \partial_2 < \tilde{\psi}_m, A_2 u > + \lambda_m^2 < (G*u),\ A_2\tilde{\psi}_m > \\ = < \tilde{\psi}_m, F > + B(\tilde{\psi}_m, (G*u))\end{aligned} \tag{63}$$

Noting the homogeneous form of the Green's identity,

$$< (G*u),\ \tilde{A}_2\tilde{\psi}_m > = < \tilde{\psi}_m, A_2(G*u) > \tag{64}$$

and that G is a function of time, it follows that

$$< (G*u), \tilde{A}_2\tilde{\psi}_m > = G* < \tilde{\psi}_m, A_2 u > \tag{65}$$

Using equation (65) in equation (63) gives

$$\partial_2 < \tilde{\psi}_m, A_2 u > + \partial < \tilde{\psi}_m, A_1 u > + (\lambda_m)^2 \; G^* < \tilde{\psi}_m, A_2 u >$$

$$= < \tilde{\psi}_m, F > + B(\tilde{\psi}_m, (G^* u)) \tag{66}$$

by using the proportionality condition in equation (30), a common inner product may be factored out of equation (66) to give

$$\ddot{\xi}_m(t) + (a + b\lambda_m^2) \; \dot{\xi}_m + (\lambda_m)^2 \; G^* \xi_m$$

$$= < \tilde{\psi}_m, F > + B(\tilde{\psi}_m, (G^* u)) + b\partial B(\tilde{\psi}_m, u) \tag{67}$$

where $\xi_m(t)$ has been defined as before in the viscous damping case in equation (32). Likewise, the generalized initial conditions, $\xi_m(0)$ and $\dot{\xi}_m(0)$ are found from this equation at $t = 0$. The coefficients $h_m(t)$ are found as before for undamped members to be

$$h_m(t) = \frac{\xi_m(t)}{Q_m} \tag{68}$$

where $\xi_m(t)$ is found from equation (67). Note that although $\xi_m(0)$ has the same form for elastic and viscoelastic members, the solution for $\xi_m(t)$ will in general be different, due to the $G^* \xi_m$ term in equation (67). These solutions coincide when G^* reduces to unity, i.e. the viscoelastic member becomes elastic.

Although a solution of the viscoelastic dynamic problem may in principle be given by equation (58), this form of the modal solution is seldom used because of its poor convergence qualities. In fact, most authors do not even consider the displacement form as a viable approach.

Therefore, attention will be focused on the more rapidly converging acceleration (or Williams) method for a general viscoelastic structural member.

The acceleration form of the modal solution may be derived by following the steps used in the previous section. Begin by rearranging equation (67) into the form,

$$G*\xi_m = -\frac{1}{(\lambda_m)^2}\left[\ddot{\xi}_m + (a+b\lambda_m^2)\,\dot{\xi}_m - H_m\right] \qquad (69)$$

where H_m is the generalized force term given by

$$H_m(t) = \int_x \tilde{\psi}_m F\,dx + B(\tilde{\psi}_m(G*u)) + b\,B(\tilde{\psi}_m,u) \qquad (70)$$

Taking the convolution of both sides of equation (58) and using equation (68) gives

$$G*u = \sum_{m=1}^{\infty} \frac{(G*\xi_m)}{Q_m}\,\psi_m(x) \qquad (71)$$

assuming the summation and convolution may be reversed. Now introduce equation (69) into (71) and after some rearrangement obtain

$$G*u = \sum_{m=1}^{\infty} \frac{\psi_m H_m}{\lambda_m^2 Q_m} - \sum_{m=1}^{\infty} \frac{\psi_m}{\lambda_m^2 Q_m}\left[\xi_m + (a+b\lambda_m^2)\,\dot{\xi}_m\right] \qquad (72)$$

The first term in equation (72) is the static contribution to the viscoelastic solution. As Valanis [3] and Robertson and Thomas [4] have pointed out, a dynamic viscoelastic problem has a solution of the form

$$u(x,t) = \bar{v}_s(x,t) + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m} \psi_m(x) \tag{73}$$

where $\bar{v}_s$ is the quasi-static viscoelastic solution. Equations (72) and (73) are equivalent by letting

$$G*\bar{v}_s = \sum_{m=1}^{\infty} \frac{\psi_m H_m}{\lambda_m^2 Q_m} \tag{74}$$

and

$$G*E_m = - \sum_{m=1}^{\infty} \frac{1}{\lambda_m^2} \left[\ddot{\xi}_m + (a+b\lambda_m^2)\, \dot{\xi}_m \right] . \tag{75}$$

Then equation (72) becomes

$$G*u = G*\bar{v}_s + \sum_{m=1}^{\infty} (G*E_m) \frac{\psi_m}{Q_m} \tag{76}$$

which implies equation (73). A more thorough understanding of the terms in equation (76) may now be undertaken. The Green's identity may be expressed in the form

$$< \tilde{\psi}_n, D(G*\bar{v}_s) > = < (G*\bar{v}_s), \tilde{\tilde{D}}\psi_n > + B(\tilde{\psi}_n, (G*\bar{v}_s)) \tag{77}$$

Substitute equations (10a) and (74) into this equation to obtain

$$< \tilde{\psi}_n, D(G^*\bar{v}_s) > = - \sum_{m=1}^{\infty} \frac{\lambda_n^2}{\lambda_m^2 Q_m} \left[H_m(t) < \psi_m, \tilde{A}_2\tilde{\psi}>_n \right] + B(\tilde{\psi}_n, (G^*\bar{v}_s)) \tag{78}$$

where integration of the series is assumed to be equal to a series of the integrals. Using the orthogonality relation for undamped systems and in view of equations (61) and (70), equation (78) becomes

$$< \tilde{\psi}_n, G^*D\bar{v}_s > = - < \tilde{\psi}_n, F > - B(\tilde{\psi}_n, (G^*u)) - b\, \partial\, B(\tilde{\psi}_n, u) - B^*(\psi_n, (G^*\bar{v}_s)) \tag{79}$$

The bilinear forms in the above equation contain contributions to the generalized forcing function from the non-homogeneous tractions on the surface. In particular, the form $B(\tilde{\psi}_n, (G^*u))$ contains the non-homogeneous surface tractions as given by equation (57c). Therefore, $B(\tilde{\psi}_n, (G^*\bar{v}_s))$ implies that the same boundary and in-span conditions apply to $\bar{v}_s$ when damping is absent ($b = 0$). In addition, a differential equation valid throughout the volume is implied by the remaining two terms in equation (79). That is, the following boundary value problem may be reduced from equation (79):

$$G^*D(\bar{v}_s(x,t)) = - F(x,t) \tag{80}$$

with boundary and in-span conditions

$$G^*L_i\bar{v}_s(x,t) = P_i(x,t) \qquad \text{on } S \tag{81}$$

which when solved will yield the quasi-static response of a viscoelastic member used to obtain the complete dynamic response as shown in equation (73). Note that this solution is quasi-static in the sense that $\bar{v}_s$ is a function of x alone which is to be found for each instant of time.

To complete the acceleration form of the solution, an uncoupled equation for $E_m(t)$ must be found. Substitute the series representation in equation (73) into the governing equation of motion, equation (57a). This gives

$$G*D\bar{v}_s + G*D\left(\sum_{m=1}^{\infty} E_m\psi_m\right) = A_1\dot{\bar{v}}_s$$
$$+ A\left(\sum_{m=1}^{\infty} \dot{E}_m\psi_m\right) + A_2\ddot{\bar{v}}_s + A_2\left(\sum_{m=1}^{\infty} \ddot{E}_m\psi_m\right) - F \quad . \tag{82}$$

where it is assumed that the infinite series converges at all points of interest. The first term in equation (82) is equal to -F which can be cancelled from either side of this equation. Premultiply by $\tilde{\psi}_n$, then integrate over the volume to obtain

$$< \tilde{\psi}_n, A_1\dot{\bar{v}}_s > + \sum_{m=1}^{\infty} \dot{E}_m < \tilde{\psi}_n, A_1\psi_m > + < \tilde{\psi}_n, A_2\ddot{\bar{v}}_s >$$
$$+ \sum_{m=1}^{\infty} \ddot{E}_m < \tilde{\psi}_n, A_2\psi_n > = \sum_{m=1}^{\infty} (G*E_m) < \tilde{\psi}_n, D\psi_m > \tag{83}$$

Note that the validity of reversing the convolution and the summation sign has been assumed. Apply equation (3) to the above equation and then use the orthogonality relation for undamped members to find

$$\ddot{E}_m(t) + (a+b\lambda_m^2)\dot{E}_m + \lambda_m^2(G*E_m) = R_m(t) \tag{84a}$$

where

$$R_m(t) = -\frac{1}{Q_m}\left[< \tilde{\psi}_m, A_1\dot{\bar{v}}_s > + < \tilde{\psi}_m, A_2\ddot{\bar{v}}_s > \right] \tag{84b}$$

and where the proportionality condition in equation (30) has been employed. Note that the condition $B(\tilde{\psi}_n, \psi_m) = 0$ has also been used. The initial condition for $E_m(t)$ may be found by premultiplying the initial condition form of equation (73) by $\tilde{\psi}_n A_2$ and then integrating over x to give

$$< \tilde{\psi}_n, A\ u(x,0) > = < \tilde{\psi}_n, A_2\bar{v}_s(x,0) > + \sum_{m=1}^{\infty} \frac{E_m(0)}{Q_m} < \tilde{\psi}_n, A_2\psi_m > \tag{85}$$

After applying undamped orthogonality, the generalized initial conditions are

$$E_m(0) = \left[< \tilde{\psi}_m, A_2u_0(x) > - < \tilde{\psi}_m, A_2\bar{v}_s(x,0) > \right] \tag{86}$$

and similarly,

$$\dot{E}_m(0) = \left[< \tilde{\psi}_m, A_2\dot{u}_0(x) > - < \tilde{\psi}_m, A_2\dot{\bar{v}}_s(x,0) > \right] \tag{87}$$

An alternative form of the right-hand side of equation (84a) may be found which incorporates the complex compliance J(t) of the viscoelastic material. With this form, either the experimental or modeling information of a given material may be easily included in the solution for $E_m(t)$. Begin by applying the proportionality condition in equation (30) to give

$$< \psi_m, A_1\dot{\bar{v}}_s > = (a+b\lambda_m^2) < \tilde{\psi}_m, A_2\dot{\bar{v}}_s > - b\ B(\tilde{\psi}_m, \dot{\bar{v}}_s) \tag{88}$$

Substitution of this condition into equation (84b) leads to

$$R_m = -\frac{1}{Q_m} \left[(c_m\partial+\partial_2) < \tilde{\psi}_m, A_2\bar{v}_s > - b\ B(\tilde{\psi}_m, \dot{\bar{v}}_s) \right] \tag{89}$$

where R_m denotes the right-hand side of equation (84a), and c_m is the proportionality constant, $(a+\lambda_m^2 b)$. The inner product may now be transformed by use of the Green's identity into

$$< \tilde{\psi}_m, A_2\bar{v}_s > = < \bar{v}_s, A_2\tilde{\psi}_m > \tag{90}$$

Using equation (10a) in equation (90) gives

$$< \tilde{\psi}_m, A_2\bar{v}_s > = -\frac{1}{\lambda_m^2} < \bar{v}_s, \bar{D}\tilde{\psi}_m > \tag{91}$$

Now apply the extended Green's identity,

$$< \bar{v}_s, \bar{D}\tilde{\psi}_m > = < \tilde{\psi}_m, D\bar{v}_s > + B(\tilde{\psi}_m, \bar{v}_s) \tag{92}$$

to obtain from equation (91) the result,

$$< \tilde{\psi}_m, A_2\bar{v}_s > = -\frac{1}{\lambda_m^2} < \tilde{\psi}_m, D\bar{v}_s' > - \frac{1}{\lambda_m^2} B(\tilde{\psi}_m, \bar{v}_s) \tag{93}$$

The term $D\bar{v}_s$ may now be transformed by taking the Laplace transform of equation (80)

$$s\bar{G}D\bar{v}_s = -\bar{F} \tag{94}$$

where the bar represents the corresponding transformed variable and

$$G*f = s\,\bar{G}\,\bar{f} \tag{95}$$

Equation (94) may be easily solved for $D\bar{\bar{v}}_s$ to obtain

$$D\bar{\bar{v}}_s = -\frac{\bar{\bar{F}}}{s\bar{G}} = -s\,\bar{J}\,\bar{\bar{F}} \tag{96}$$

where the relation between the Laplace transforms of the complex compliance $\bar{J}$ and the relaxation modulus $\bar{G}$ has been used. Taking the inverse Laplace transform of equation (96) gives

$$D\bar{v}_s = -J * F \tag{97}$$

Finally, substitute this equation back into equation (93), which when placed in equation (89) leads to

$$R_m = -\frac{1}{Q_m}\left\{\frac{(c_m\partial+\partial_2)}{\lambda_m^2}\left[<\tilde{\psi}_m, J * F> - B(\tilde{\psi}_m,\bar{v}_s)\right] - b\,B(\tilde{\psi}_m,\dot{\bar{v}}_s)\right\} \tag{98}$$

This is the alternate form for the right-hand side of equation (84a) which was sought. This form of $R_m(t)$ involves directly the complex compliance J of the viscoelastic material. When substituted back into equation (84a) the solution for $E_m(t)$ may now be found by taking several approaches. If a known viscoelastic model composed of springs and dashpots has been selected, the compliance will be known once the model parameters have been defined. These parameters may be prescribed a priori or, chosen by fitting experimental data to the model [14]. In addition, experimental data may be used directly in equation (98) without

any reliance on a particular model [15], in which case equation (84a) may be solved by Laplace transforms, which may be inverted analytically for simple models. An example of how this procedure may be carried out for a Voigt-Kelvin material will be shown subsequently. Otherwise, a numerical inversion process may be needed. Another approach which may be taken in finding the coordinates $E_m(t)$ from equation (84a) has been suggested by Valanis [3]. When a Laplace transform of equation (84a) has been taken, a Voltera integral equation of the second kind results. Then, any number of techniques may be used to solve this integral equation. Note that if the usual normal mode approach had been undertaken, the above comments on how to find $E_m(t)$ may be applied unchanged in determining $\xi_m(t)$ in equation (67). However, as mentioned previously the acceleration method's more rapidly convergent solution makes solving the additional quasi-static problem worth the trouble, since $\bar{v}_s$ may be found easily for many structural members by using a correspondence principle.

To summarize breifly, the dynamic response of a viscoelastic structural member on a proportional viscous foundation may be formally expressed by

$$u(x,t) = \bar{v}_s\ (x,t) = \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m}\ \psi_m(x) \tag{99}$$

where $\psi_m(x)$ and Q_m are the corresponding undamped elastic member eigenfunctions and norm, respectively. The temporal coefficients may be found by solving

$$\ddot{E}_m(t) + (a+b\lambda_m^2)\dot{E}_m + \lambda_m^2(G{*}E_m) = R_m(t) \tag{100}$$

and $R_m(t)$ may take on either of the two forms as given by equations (84b), or (98). As special cases, when $a = b = 0$, a viscoelastic member without a damped foundation may be examined, and when the bilinear form is absent the member possesses homogeneous boundary and in-span conditions.

Forced Response of a Structural Member with Linear Hysteretic Damping

In this section, the concept of hysteretic damping will be explored for the purpose of finding the forced response of structurally damped structural members. The hysteretic damping to be considered is linear; hence, the standard assumed solution of modal analysis will again be employed. Both self-adjoint as well as non-self-adjoint systems will again be considered.

Consider a continuous structural member with linear hysteretic damping governed by the following equation of motion:

$$Du(x,t) = \sum_{j=1}^{2} A_j(x)\partial_j u(x,t) - f(x)e^{i\Omega t} \tag{101a}$$

with the boundary and in-span conditions,

$$L_i u(x,t) = P_i(x)e^{i\Omega t} \qquad \text{on } S \tag{101b}$$

where $f(x)$ and $P_i(x)$ are vectors of functions which represent the spatial variations of the forcing function, and boundary and in-span conditions, respectively. Note that the temporal portions of the excitation functions are harmonically oscillating with a driving

frequency Ω, which is a simplification of $F(x,t)$ given by equation (1a). The initial conditions are zero because interest is focused on the steady state response, or the particular solution of the differential equation.

Based upon the form of the forcing function and boundary conditions, the steady state response sought will also be harmonic with frequency Ω. This is identical to a single degree of freedom system, since all points in the member are oscillating in phase. Hence

$$\frac{\partial u}{\partial t}(u,t) = \partial u = i\Omega u(x,t) \tag{102}$$

which when substituted into Equation (101a) yields

$$Du(x,t) = i\Omega A_1 u + A_2 \partial_2 u - f(x)e^{i\Omega t} \tag{103}$$

This matrix equation represents the governing equations of motion of a structurally damped member undergoing steady state motion. As before, the goal is to determine the proper modal expansion which gives the response. This entails defining the modes and orthogonality condition, and the uncoupling coordinates. It will be shown that only the undamped modes are needed to resolve the solution so that the response may be written as

$$u(x,t) = \sum_{m=1}^{\infty} q_m(t)\psi_m(x) \tag{104}$$

Hence, the coordinates $q_m(t)$ must be determined to complete the steady state response.

Once again the starting point of the general formulation is the extended Green's identity,

$$< \tilde{\psi}_m, Du > - < u, \tilde{D}\tilde{\psi}_m > = B(\tilde{\psi}_m, u) \tag{105}$$

Since A_1 and A_2 are restricted to be only functions of x, (or at best, spatial matrix differential operators with homogeneous boundary conditions) $B(\tilde{\psi}_m, u)$ contains all the elastic non-homogeneous boundary and in-span conditions of the member. Substitute the classical free vibration problem (equation (3a)) and equation (103) into the above identity and after some rearrangement obtain

$$\begin{aligned} &\partial_2 < \tilde{\psi}_m, A_2 u < + i\Omega > \tilde{\psi}_m, A_1 u > + \lambda_m^2 < \tilde{\psi}_m, A_2 u > \\ &\qquad = < \tilde{\psi}_m, f(x)e^{i\Omega t} > + B(\tilde{\psi}_m, u) \end{aligned} \tag{106}$$

where the classical form of equation (25) has been used. Note that the imaginary unit i makes the operator complex. However, by foregoing the Hermitian norm the Green's identity remains the same for both real and complex operators. That is, the imaginary unit is effectively treated as a constant. By defining $\xi_m(t)$ as previously (see equation (32), the above equation may be rewritten as

$$\begin{aligned} &\ddot{\xi}_m(t) + \lambda_m^2 \xi_m + i\Omega < \tilde{\psi}_m, A_1 u > \\ &\qquad = < \tilde{\psi}_m, f(x)e^{i\Omega t} > + B(\tilde{\psi}_m, u) \end{aligned} \tag{107}$$

This expression is not yet uncoupled due to the A_1 term. A proportionality condition must be invoked, which for hysteretic damping, may be called proportional structural damping. Recall that for a single degree of freedom system, an equivalent viscous damping coefficient can be defined for hysteretic damping which is proportional to the stiffness and inversely proportional to the driving frequency. For a continuous member, a similar equivalence may be established so that

$$A_1 u = -\frac{\gamma}{\Omega} Du \tag{108}$$

where Meirovitch refers to γ as the structural damping factor [16]. Using this condition leads to

$$\begin{aligned} i\Omega < \tilde{\psi}_m, A_1 u > &= -i\gamma < \tilde{\psi}_m, Du > = -i\gamma < u, \tilde{D}\tilde{\psi}_m > \\ &\quad -i\gamma\, B(\tilde{\psi}_m, u) \end{aligned} \tag{109}$$

which after employing classical free vibration gives

$$\begin{aligned} i\Omega < \tilde{\psi}_m, A_1 u > &= i\lambda_m^2\gamma < u, \tilde{A}_2\tilde{\psi}_m > -i\gamma B(\tilde{\psi}_m u) \\ &= i\lambda_m^2\gamma < \tilde{\psi}_m, A_2 u > - i\gamma B(\tilde{\psi}_m, u) \end{aligned} \tag{110}$$

Thus equation (107) becomes

$$\begin{aligned} \dot{\xi}_m(t) + \lambda_m^2(1 + i\gamma)\, \xi_m &= e^{i\Omega t} < \tilde{\psi}_m, f(x) > \\ &\quad + (1 + i\gamma)\, B\,(\tilde{\psi}_m, u) \end{aligned} \tag{111}$$

This equation is identical in form to an uncoupled equation for a SDOF system with hysteretic damping. The solution for non-homogeneous (time-dependent or constant) boundary and in-span conditions may be found easily. For example, if the boundary and in-span conditions are separable such that the right-hand side of the above equation may be written as

$$f_m e^{i\Omega t} = [< \tilde{\psi}_m, f(x) > + (1 + i\gamma)\ B(\tilde{\psi}_m, U(x)]\ e^{i\Omega t} \tag{112}$$

the solution for $\xi_m(t)$ is given by $= \dfrac{f_m e^{i\Omega t}}{\lambda_m^2(1 + i\gamma) - \Omega^2}$ (113)

This is the particular solution of equation (111) since it is assumed that all initial condition transients have died out. Since classical orthogonality was used, $q_m(t)$ in equation (104) may be shown to be

$$q_m(t) = \frac{\xi_m(t)}{Q_m} \tag{114}$$

so that the steady state response may be readily found. This result is identical to that obtained by Meirovitch [16], except that this formulation now includes non-self-adjoint systems, non-homogeneous boundary and in-span conditions, as well as incorporating the very powerful matrix differential operator notation.

Applications:

In this section will be presented several applications of the general dynamic theory to typical damped structural members. One objective of presenting these examples is to illustrate in detail usage of the general formulas provided in previous chapters. However, these

formulas provide the dynamic response in terms of some quantities which have been assumed to be known. For example, the solution of the damped free vibration problem was not actually solved but merely represented symbolically by the vector of eigenfunctions v_n and the corresponding eigenvalue S_n. Hence, another goal of these examples is to demonstrate how the response may be calculated for some simple damped structural members. It will be shown that transfer matrices may be easily coupled to a modal analysis of structural members which possess one independent coordinate. The dynamic response is computed for two typical members, a beam on a non-proportional viscous foundation, and a viscoelastic beam. Since no similar results could be found in the literature, these cases will be presented as benchmark examples. For more complicated members, the response will be theoretically formulated and then compared with available theoretical results. The intention here is to show how the general formulas presented in this section lead to the dynamic response without the involved algebraic manipulations taken in the past. Whether the applications are numerical or theoretical, these examples are intended to be illustrative rather than comprehensive. Therefore, complexities in a structural member which may be treated without confusion in theory but are tedious in practice, will be avoided to demonstrate clearly the methodology of finding the dynamic response.

Cantilever Beam on a Damped Viscous Foundation

Consider the uniform cantilever beam shown in Figure 1, which is resting on a stepped damped foundation. The value of the mass density

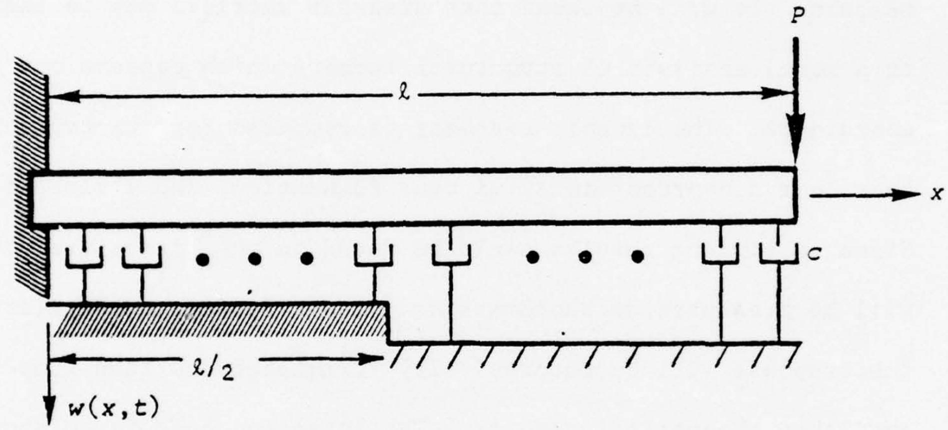

Figure 1 Damped Cantilever Beam

(ρ), modulus of elasticity (E), moment of inertia (I), length (ℓ), and viscous foundation coefficient (c) are

$$\begin{aligned} \rho &= .002301 \text{ lb - sec}^2/\text{in}^2 \\ E &= 10 \times 10^6 \text{ psi} \\ I &= .7854 \text{ in}^4 \\ \ell &= 100 \text{ in} \\ c &= \begin{cases} .0011505 \text{ lb-sec/in}^2 & \text{for } x < 50 \text{ in} \\ .0017257 \text{ lb-sec/in}^2 & \text{for } x > 50 \text{ in} \end{cases} \end{aligned} \tag{115}$$

The force P applied at x=100 in. is a static load of 20 lbs. At $t = 0^+$ it will be removed and the previous formulas will be used to find the resulting transient response.

The equations of motion governing this member have been introduced in equations (1). The matrix differential operators are given by

$$D = \begin{bmatrix} 0 & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial x} & 1 \\ 0 & \frac{\partial}{\partial x} & -\frac{1}{EI} & 0 \\ \frac{\partial}{\partial x} & 1 & 0 & 0 \end{bmatrix} \tag{116a}$$

$$A_1 = \begin{bmatrix} -c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{116b}$$

$$u = \begin{Bmatrix} w \\ \theta \\ M \\ V \end{Bmatrix} \tag{116c}$$

In addition, the forcing function F(x,t) is set equal to zero and the initial condition vectors are

$$u_o(x) = \begin{Bmatrix} w_0 \\ \theta_0 \\ 0 \\ 0 \end{Bmatrix} \qquad \dot{u}_o(x) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{117}$$

where w_0 and θ_0 are the static deflection and rotation due to P given by

$$w_0 = \frac{Px^2}{2EI} (\ell - x/3) \tag{118}$$

$$\theta_0 = \frac{Px}{EI} (\frac{x}{2} - \ell) \tag{119}$$

Note that the proportionality condition given by equation (30) is not satisfied by the variable damped foundation. However, this stepped foundation does not intefere with the self-adjoint property of this member. Therefore, the damped adjoint eigenfunctions $\tilde{v}_m$ are not needed to determine the response. For this case, use of equations (22), (37), and (45b) leads to the deflection response.

$$w(x,t) = \frac{P}{2EI} \sum_{m=1}^{\infty} \left\{ \frac{1}{2S_m\rho \int_0^{\ell} (w_m)^2 dx + \int_0^{\ell} c(w_m)^2 dx} \left[e^{S_m t} \right.\right.$$

$$\left.\left. \int_0^{\ell} (c+S_m\rho)(x^2\ell - x^3/3)\ dx \right]\right\} w_m(x) \tag{120}$$

In this expression S_m is the complex eigenvalue and w_m is the deflected mode shape, both of which are found from the damped free vibration problem. This free motion problem may not be solved analytically. Rather, a numerical technique must be employed. Since this beam possesses one independent coordinate direction, transfer matrices may be used. For further simplicity, approximate the continuous member by dividing it into massless elastic sections punctuated by mass lumps (stations). The damped foundation is also concentrated into discrete dashpots located at the mass stations. For example, this leads to the equation,

$$\begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i^R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S^2 m_i + Sc_i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i \tag{121}$$

or,

$$\begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i^R = [T_M]_i \begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i^L \tag{122}$$

where the superscripts R and L refer to locations just to the right and left, respectively, of the mass lump (m_i) and dashpot (c_i) at $x = x_i$. The transfer matrix corresponding to the mass and damping lump at $x = x_i$, $[T_M]_i$, allows the state variables just to the right of the mass station to be known in terms of those just to the left. In a similar way, the expression for an elastic section is given by

$$\begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_{i+1}^L = \begin{bmatrix} 1 & -\Delta\ell_i & \frac{-(\Delta\ell_i)^2}{2E_iI_i} & \frac{-(\Delta\ell_i)^3}{6E_iI_i} \\ 0 & 1 & \frac{\Delta\ell_i}{E_iI_i} & \frac{(\Delta\ell_i)^2}{2E_iI_i} \\ 0 & 0 & 1 & \Delta\ell_i \\ v & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i^R \tag{123}$$

or,

$$\begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_{i+1}^L = [T_E]_i \begin{bmatrix} w \\ \theta \\ M \\ V \end{bmatrix}_i^L \tag{124}$$

where $\Delta\ell_i$, E_i, and I_i are the length, modulus of elasticity, and moment of inertia, respectively, of the elastic section from x_i^R to x_{i+1}^L.

The matrix $[T_E]_i$ transfers the known state variables at x_i to state variables at x_{i+1}. With equations (122) and (124) the response in terms of the deflection, slope, moment, and shear at any location may be found in terms of the known values at x = 0 by merely alternately multiplying mass and elastic transfer matrices down the beam. It is obvious that variations in the mass density, damping foundation modulus, and moment of inertia can be easily accounted for by merely changing m_i, c_i, E_i, and I_i for each transfer matrix. Other effects such as rotary inertia, shear deformation, in-span conditions, and an elastic foundation may also be readily included by introducing similar transfer matrices. Such catalogues of transfer matrices, including a more thorough treatment of this approach, may be found in several sources such as Pestel and Leckie [17], and Pilkey [18]. A listing of the damped frequencies found for this example are given in Table 1. Using the damped mode shapes found above, the response may be computed by using equation (120). This response is shown in Figure 2.

In conclusion, a simple Euler-Bernoulli beam has been presented to demonstrate how easily the response may be found using the general formulas presented in this work. The response may be found for more complicated loading conditions, and non-homogeneous boundary and in-span conditions by using the same equations. The approach is identical to that taken for this simple beam, only the details become a bit more tedious. In addition, complications such as shear deformation and rotary inertia are also readily included in the previous analysis. This is done by modifying the appropriate transfer matrices in the damped free motion problem and by altering the matrices A_1 and A_2 used in the general formulas.

Table 1

Complex Frequencies for a Damped Beam with Variable Foundation

n	S_n	
	$Re\{S_n\}$ (1/SEC)	$Im\{S_n\}$ (CPS)
1,2	$-3.6869335 \times 10^{-1}$	± 3.2663992
3,4	$-3.2006817 \times 10^{-1}$	$\pm 2.0436355 \times 10^{1}$
5,6	$-3.1172523 \times 10^{-1}$	$\pm 5.7129133 \times 10^{1}$
7,8	$-3.1291877 \times 10^{-1}$	$\pm 1.1176201 \times 10^{2}$
9,10	$-3.1274025 \times 10^{-1}$	$\pm 1.8443682 \times 10^{2}$

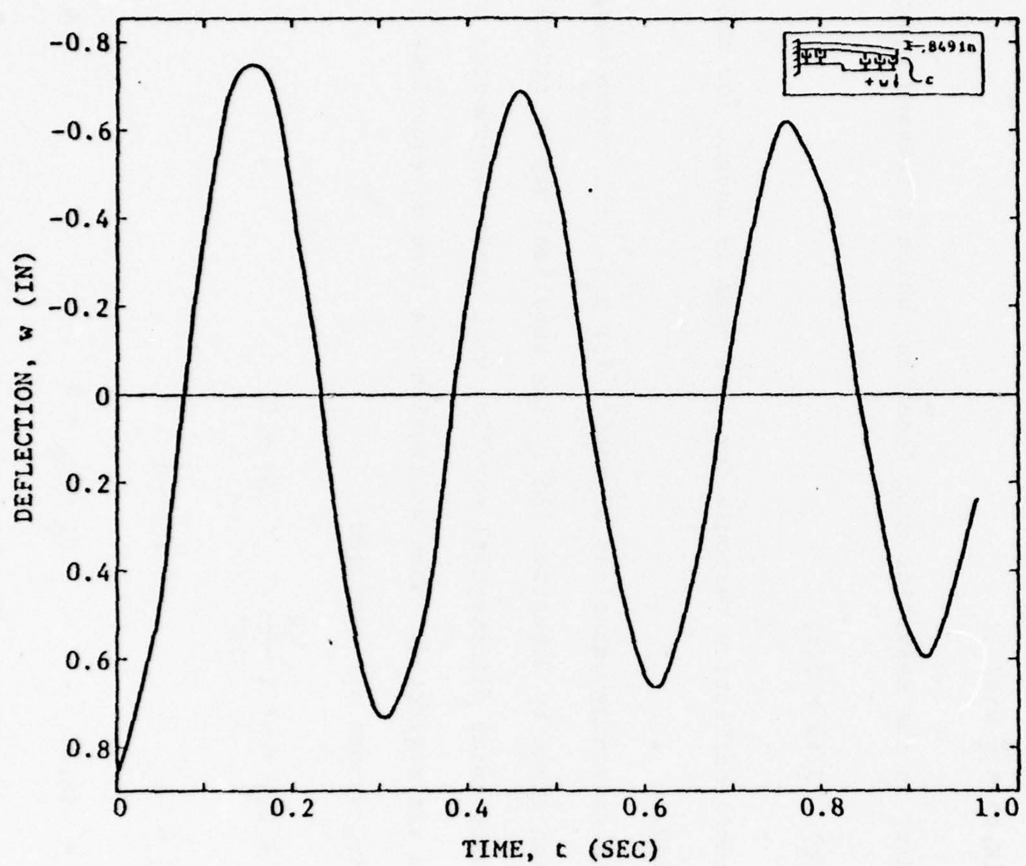

Figure 2 Transverse Tip (x = 100 in) Deflection of Beam on a Variable Viscous Foundation

Dynamic Response of a Voigt-Kelvin Beam

Consider the undamped Euler-Bernoulli beam shown in Figure 3, which is composed of a Voigt-Kelvin viscoelastic material. Assume that the linearly varying distributed load shown in Figure 3 is applied at $t = 0$ and then maintained at that value for all future time. That is,

$$q(x,t) = \frac{x}{\ell} H(t) \tag{125}$$

where $H(t)$ is the Heaviside unit function, which possesses the property

$$H(t) * J(t) = J(t) \tag{126}$$

The dynamic deflection response for this beam is sought for this loading condition.

The governing equations of motion for a general viscoelastic beam have been given by Robertson [15]. The equations may be reduced to one fourth order differential equation when shear deformation and rotary inertia are neglected. When written in the form of equations (57), the following values are assigned:

$$D = 2(1 = \nu) I \frac{\partial^4}{\partial x^4} \qquad A_1 = 0 \tag{127a}$$

$$A_2 = -\rho \qquad F = -q \tag{127b}$$

$$u = w(x,t) \qquad u_0 = \dot{u}_0 = 0 \tag{127c}$$

$$(L_1)_{x=0,\ell} = 1 \qquad (L_2)_{x=0,\ell} = \frac{\partial^2}{\partial x^2} \tag{127d}$$

$$(P_1)_{x=0,\ell} = (P_2)_{x=0,\ell} = 0 \tag{127e}$$

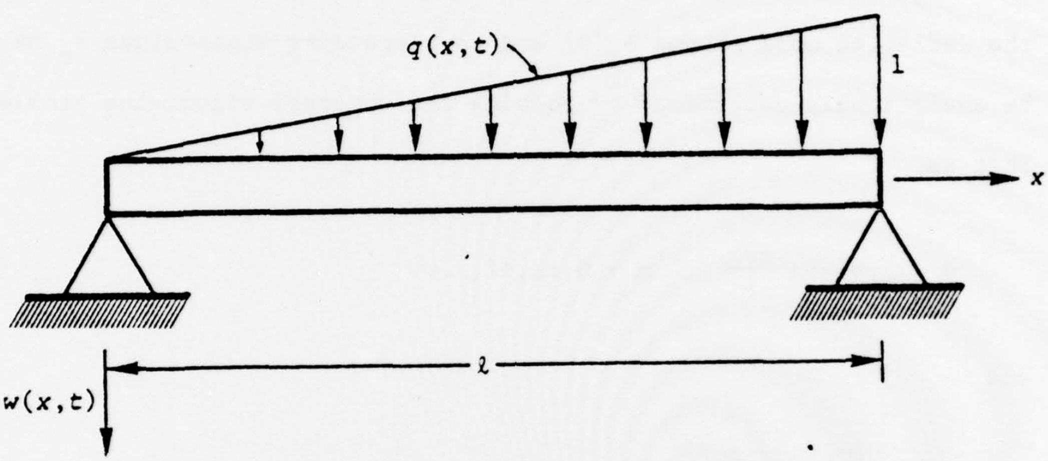

Figure 3 Voigt-Kelvin Beam Under Variable Loading

The dynamic response of this viscoelastic beam may be found from equation (73) to be

$$w(x,t) = w_s(x,t) + \sum_{m=1}^{\infty} \frac{E_m(t)}{Q_m} \psi_m(x) \tag{128}$$

where $w_s(x,t)$ is the quasi-static deflection. For this simple member, the deflected mode shapes $\psi_m(x)$ and corresponding eigenvalues λ_m may be analytically determined by solving the undamped eigenvalue problem. That is,

$$\psi_m(x) = \sin \frac{m\pi x}{\ell} \qquad m = 0, \pm 1, \pm 2, \ldots \tag{129}$$

and

$$\lambda_m = \left(\frac{m\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}} \tag{130}$$

so that the corresponding norm may be explicitly determined to be

$$Q_m = \rho\ell/2 \tag{131}$$

Note that since the elastic beam is self-adjoint, only the modes $\psi_m(x)$ are needed to determine the response. Therefore, in the series portion of equation (128), only the temporal coefficient remains to be determined. To find $E_m(t)$, recall from equations (84a) and (98) that

$$\ddot{E}_m(t) + \lambda_m^2 G * E_m = \frac{1}{\lambda_m^2 Q_m} \frac{\partial^2}{\partial t^2} \left[\int_0^{\ell} \psi_m (J*q)\, dx \right] \tag{132}$$

where a = b = 0 since the beam is undamped. In addition, the initial values, $E_m(0)$ and $\dot{E}_m(0)$ given by equations (86) and (87), are zero because the initial conditions u_0 and $\dot{u}_0$ are zero. Introduce equations (125), (129), (130), and (131) into the above expression to arrive at

$$\ddot{E}_m + \lambda_m^2 G{*}E_m = \left(\frac{-2(-1)^m \ell^4}{(m\pi)^5 EI}\right)\ddot{J}(t) \tag{133}$$

where equation (126) has been utilized. Eliminate the time variable in this expression by taking the Laplace transform to obtain

$$s^2\bar{E}_m + \lambda_m^2 s\bar{G}\,\bar{E}_M = -\left(\frac{2(-1)^m \ell^4}{(m\pi)^5 EI}\right) s^2\bar{J} \tag{134}$$

where the initial conditions of the complex compliance vanish since they are proportional to $E_m(0)$ and $\dot{E}_m(0)$. Solve for the Laplace transform of $E_m(t)$ (i.e. $\bar{E}_m(s)$) to obtain

$$\bar{E}_m = \frac{-s^3 JJL_m}{(\lambda_m^2 + s^3\bar{J})} \tag{135}$$

where for convenience

$$L_m = \frac{2(-1)^m \ell^4}{(m\pi)^5 EI} \tag{136}$$

The Laplace transform of the complex compliance for a Voigt-Kelvin material [19] is given by

$$\bar{J} = \frac{1}{q_0}\left(\frac{1}{s} - \frac{1}{(\kappa + s)}\right) \tag{137}$$

where

$$\kappa = q_0/q_1 \tag{138}$$

and q_0 and q_1 are the spring and dashpot constants in the Voigt-Kelvin model. Introduce equation (137) into equation (135), and after some rearrangement arrive at the expression

$$\bar{E}_m = - L_m \left\{\frac{s}{(s-r_1)(s-r_2)} + \frac{\lambda_m^2}{\kappa}\left[\frac{1}{(s-r_1)(s-r_2)}\right] - \frac{1}{\kappa^2}\left[\frac{1}{s+\kappa}\right]\right\} \tag{139}$$

where

$$r_1, r_2 = \frac{-\lambda_m^2 q_1}{2} \pm \frac{\lambda_m}{2}\sqrt{q_1^2 \lambda_m^2 - 4q_0} \tag{140}$$

and where the method of partial fractions has been utilized. Using a table of Laplace transforms, $E_m(t)$ *may be found by inverting equation* (139) which leads to

$$\begin{aligned} E_m(t) = -L_m \Bigg[& \frac{1}{\lambda_m \sqrt{\lambda_m^2 q_1^2 - 4q_o}} \left(r_1 e^{r_1 t} - r_2 e^{r_2 t}\right) \\ & - \frac{\lambda_m}{\kappa \sqrt{\lambda_m^2 q_1^2 - 4q_o}} \left(e^{r_2 t} - e^{r_1 t}\right) - \frac{1}{\kappa^2} e^{-\kappa t} \Bigg] \end{aligned} \tag{141}$$

This is the expression which was sought for $E_m(t)$. Since it can be evaluated from this equation, all the ingredients needed for the series portion of the solution in equation (128) have been found in closed form. To determine the quasi-static part of the response in this equation, make use of a so-called correspondence principle. That is, as

Flugg [20] explains, the static deflection of a viscoelastic member may be found by finding the deflection of the corresponding elastic member and then replacing the modulus of elasticity (E) by the ratio of polynomials $\bar{Q}(s)/\bar{P}(s)$. Here $\bar{Q}(s)$ and $\bar{P}(s)$ are the Laplace transforms of the temporal differential operators P and Q, introduced in equations (135). In general, to find the quasi-static response $w_s(x,t)$, utilize this correspondence principle for each time step. For this case it will be shown that this principle need only be applied once, since the load is constant for all time. From elementary strength of materials, the static deflection of an elastic simply supported beam is

$$w(x) = \frac{q}{180\ EI}\,[3x^4 - 5x^2\ell^2 + 2\ell^4] \tag{142}$$

Now replace q and E by $(x/\ell)\ \frac{1}{s}$ and $\bar{Q}(s)/\bar{P}(s)$, respectively, to obtain

$$\bar{w}_s(x,s) = \frac{1}{180\ I\ell q_1}\left(\frac{1}{s(\kappa + s)}\right)[3x^5 - 5x^3\ell^2 + 2x\ell^4] \tag{143}$$

which is the Laplace transform of the quasi-static deflection. Note that for a Voigt-Kelvin solid [19],

$$\frac{\bar{Q}(s)}{\bar{P}(s)} = q_1(\kappa + s) \tag{144}$$

equation (143) may be easily inverted to give for the quasi-static deflection response,

$$w_s(x,t) = \left(\frac{3x^5 - 5x^3\ell^2 + 2x\ell^4}{180\ I\ell q_0}\right)(1 - e^{-\kappa t}) \tag{145}$$

which decays as time grows since $\kappa > 0$. Hence, the solution of a Voigt-Kelvin beam given by equation (128) has been determined. It is apparent that many other methods of solution of equation (132) could have been pursued. For example, had $J(t)$ been experimentally determined, a numerical solution of equation (132) would have been attempted. Or, this experimental data could be fitted to a more complicated but more realistic model. However, for more complicated viscoelastic models, a drawback to the previous method of solution is that an inverse Laplace transform of a complicated function has to be taken. In most cases of this sort, a numerical inverse Laplace transform would probably be warranted.

Summary

It has been shown in this work that a general theory for the dynamic response of linear damped continuous structural members may be formulated using a modal analysis. A general set of formulas was derived to provide the ingredients needed to construct the dynamic response. The ingredients included a statement of the free vibration problem and a determination of the orthogonality relation and the uncoupled temporal coefficients. As shown in the previous chapters, these elements of a modal solution were found in terms of the (assumed known) governing equations of the member. Therefore, the dynamic response can be explicitly written out by merely inspecting the governing equations. And since these formulas apply to self-adjoint as well as non-self-adjoint structural members with or without non-homogeneous boundary and in-span conditions, the dynamic response of the most general

type of structural member may be found. When uncoupling was not possible, another advantage of this general approach became apparent by exposing the mathematical limitations imposed by the physical model.

This work was divided into three distinct sections, depending on the type of linear damping present in the structural member. Viscous, viscoelastic, and hysteretic damping were treated. The formulas derived for the ingredients in a modal expansion are summarized in Table 2. All the formulas needed for each form of damping are not explicitly shown for the sake of clarity. The lengthier formulas are only referenced in this table, where parentheses denote the primary equations for the terms in the dynamic response. As can be seen from this table, knowing only the solution of the free vibration problem (i.e., ψ_n and λ_n or v_n and S_n) the dynamic response may be readily determined. These formulas are applicable to damped (or undamped) self-adjoint and non-self-adjoint structural members with arbitrary loading, non-homogeneous boundary and in-span conditions, and initial conditions. In all these cases, the general formulas are applied in exactly the same way to find the dynamic response.

In summary, a comprehensive theory has been presented for the dynamic response of linear continuous damped structural members. The formulation depends only on the equations of motion and is independent of the specific structural member. Hence, the dynamic response of a wide class of linear structural members may be determined by the same theory.

Table 2 Formulas for the Dynamic Response

Classification	Dynamic Response	Temporal Coefficients	Norm	Quasi-Static Response
I. Proportional Viscous Damping				
a) Displacement Method	$u(x,t) = \sum_{m=1}^{\infty} \frac{\xi_m(t)}{Q_m} \psi_m(x)$	$\xi_m(t)$: (33a)	Q_m: (20)	
b) Acceleration Method	$u(x,t) = v_s^{\dagger} + \sum_m \frac{C_m^{\dagger}(t)}{Q_m} \psi_m(x)$	$C_m(t)$: (51)	"	$v_s^{\dagger}(x,t) = \sum_{m=1}^{\infty} \frac{\psi_m F_m^{\dagger}}{\lambda_m^2 Q_m}$
II. Non-Proportional Viscous Damping				
a) Displacement Method	$u(x,t) = \sum_m \frac{\eta_m(t)}{N_m} v_m(x)$	$\eta_m(t)$: (37)	N_m: (22)	
b) Acceleration Method	$u(x,t) = v_s + \sum_m \frac{C_m(t)}{N_m} v_m(x)$	$C_m(t)$: (56)	"	$v_s(x,t) = -\sum_{m=1}^{\infty} \frac{v_m F_m}{S_m N_m}$

Table 2 (Continued)

Classification	Dynamic Response	Temporal Coefficients	Norm	Quasi-Static Response
III. Viscoelastic Material Hereditary Integral	$u(x,t) = \bar{v}_s + \sum_m \frac{E_m(t)}{Q_m} \psi_m(x)$	$E_m(t)$: (84a) 84b) or (84a) (98)	Q_m: (20	$\bar{v}_s(x,t)$: (80)
IV. Hysteretic Damping	$u(x,t) = \sum_m \frac{\xi_m(t)}{Q_m} \psi_m(x)$	$\xi_m(t)$: (11) or (12) (13)	"	

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